Mackey Closure Operators and Locally Convex Topologies in Linear Orthogonality Spaces

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Abstract
This paper brings to the fore characterizations and applications of Mackey closure operators and locally convex topologies. It shows that the orthogonality relation arising from a non-degenerate orthosymmetric sesquilinear form on a vector space yields in a natural way a Mackey closure operator. Extensions and generalizations of some known results are also shown.

KEY WORDS: Closure Operator, Locally Convex Topologies, Orthogonality.

Introduction
Let \((V, \leq)\) be a partially ordered set with \(T: V \rightarrow V\). Then \(T\) is called a closure operator on \(V\) if and only if (i) \(e \leq f\) implies \(T(e) \leq T(f)\), (ii) \(T = T^2\), (iii) \(e \leq T(e)\) for all \(e \in V\). The class of closure operators, called Mackey closure operators, was motivated by the work of G.W. Mackey (1995). Given a lattice \((L, +, \land)\) with 0, a map \(T : L \rightarrow L\) is called a Mackey closure operator on \(L\) if and only if (i) \(T\) is a closure operator on \(L\) and (ii) if \(x\) is \(T\)-closed and \(p\) is an atom of \(L\) then \(x + p\) is \(T\)-closed. The notion of a linear orthogonality relation was introduced by R. Piziak and H.R. Fischer in a paper titled Order Structure associated to quadratic spaces. It was introduced to give a uniform approach to the “geometry” of sesquilinear forms. A vector space \(E = (E, \perp)\) is called a linear orthogonality space, if \(\perp\) is a binary relation on \(E\) such that (i) \(x\perp y\) if and only if \(y\perp x\), (ii) \(\{x\}\perp \{y\} = \{x \in E : y \perp x\}\) is a subspace for all \(x \in E\), (iii) \(x\perp y\) for all \(y\) implies \(x = 0\). R. Piziak, (1997) showed that the orthogonality relation arising from a non-degenerate orthosymmetric sesquilinear form on a vector space yields in a natural way a Mackey closure operator. Hence, he gave characterization of those orthogonality relations on a vector space arising from such forms. This provided a generalization to infinite dimensions of a well-known result of (Birkhoff and Von Neumann, 1936). Kakol and Sorjomen, (1991) showed that if \((E, \perp)\) has a vector space topology, which gives the same closed subspaces as the linear orthogonality relation, then there exists always a locally convex topology with the same property. Those topologies which give the same closed subspaces as the linear orthogonality relation were characterized with the aid of Hahn Banach property. This gave a necessary and sufficient condition for a topology to be locally convex. As a by-product, we have an extension of a theorem of (Kakurtani and Mackey, 1944, 1946).

Closure Operators
We state the lattice-theoretic version of Mackey’s characterization of dual modular pairs and closed sums in the lattice of closed subspaces of a real normed linear space. The orthogonality relation arising from a non-degenerate orthosymmetricsesquilinear form on a vector space yields in a natural way a Mackey closure operator. With the characterization of those orthogonality relations on a vector space arising from such forms, we present a generalization to infinite dimensions of a well-known result of (Birkhoff and von Neumann, 1936).

**Basic Definitions**

**Definition 1:** Let \((V, \leq)\) be a partially ordered set with \(T : V \to V\). Then \(T\) is called a closure operator on \(V\) if and only if

1. \(e \leq f\) implies \(T(e) \leq T(f)\),
2. \(T = T^2\)
3. \(e \leq T(e)\) for all \(e \in V\).

\(T(e)\) is the \(T\)-closure of \(e\). If \(T(e) = e\), we say that \(e\) is \(T\)-closed. Let \(V_T\) denote the set of all \(T\)-closed elements of \(V\) also ordered by \(\leq\). If \(V\) has an order zero, we say that \(T\) is normalized provided \(T(0) = 0\).

**Theorem 1** (Everett, 1944): Let \(T\) be a closure operator on \(V\). Then:

1. any maximal element of \(V\) is \(T\)-closed;
2. \(T(e)\) is the smallest \(T\)-closed element of \(V\) containing \(e\); \(f\)
3. if \(e_a \in V_T\) and the infimum \(\cap e_x\) exists in \(V\), then \(\cap e_a\) is in \(V_T\) and is the infimum of \(\{e_a\}\) relative to \(V_T\);
4. if \(e_a \in V_T\) and the supremum \(\cup e_a\) exists in \(V\), then \(T(\cup e_a)\) in
5. \(V_T\) is the supremum of \(\{e_a\}\) relative to \(V_T\);
6. for any \(e \in V\), \(T(e) = \cap \{f \in V_{\geq f} | e \leq f\}\).

**Definition 2:** Let \((L, +, \land)\) be a lattice with zero 0. A triple \((e, f, g)\) of elements of \(L\) is called a distributive triple if and only if

1. \(e \land (f + g) = (e \land f) + (e \land g)\),
2. \(f \land (e + g) = (f \land e) + (f \land g)\),
3. \(g \land (e + f) = (g \land e) + (g \land f)\).

A pair \((e, f)\) is called a modular pair, \(\mu(e, f)\), if and only if \(x \leq f\) implies \(x + (e \land f) = (x + e) \land f\).

The lattice \(L\) as the covering property if and only if \(p\) an atom and \(a \land p = 0\) implies \(a + p\) covers \(a\).

The Lattice \(L\) is said to possess the atomic exchange property if and only if whenever \(p\) and \(q\) are atoms and \(q \leq b\), then \(q \leq b + p\) implies \(p \leq b + q\).

**Mackey Closure Operators**

Let \((L, +, \land)\) denote a lattice with zero 0. If \(T\) is a closure operator on \(L\), \(\cup\) denotes the supremum operation in \(L_T\).

**Definition 3:** A map \(T : L \to L\) is a Mackey closure operator on \(L\) if and only if

1. \(T\) is a closure operator on \(L\)
2. if \(x\) is \(T\)-closed and \(p\) is an atom of \(L\) then \(x + p\) is \(T\)-closed.

**Lemma 1** (Ore, 1944): If \(T\) be a normalized Mackey closure operator on \(L\), then any atom in \(L\) is \(T\)-closed. In fact, any finite joint of atoms is \(T\)-closed.

**Lemma 2** (Ore, 1944): Let \(T\) be a closure operator on \((L, +, \land)\). Let \(e\) and \(f\) be \(T\)-closed and suppose \(\mu^*(e, f)\) in \(L\). Then if \(e + f\) is \(T\)-closed, we must have \(\mu^*(e, f)\) in \(L_T\).

**Proof:**

Let \(x\) be closed with \(x \geq f\). We show \((x \land e) \lor f \geq x \land (e \lor f)\) in \(L_T\).
But $x\Lambda(e+g) = (x \Lambda e) f \leq T((x \Lambda e)+f) = (x \Lambda e) \forall f$ using (Lemma 1 (v)) and the fact that $x \Lambda e$ and $f$ are closed.

**Lemma 3** (Ore, 1944): Let $(L,+,\Lambda)$ be an atomistic lattice with covering property (i.e. AC lattice) with $T$ a Mackey closure operator on $L$. Let $e$ and $f$ be $T$-closed. Let $p$ be an atom with $p \leq e \forall f \in L_T$ but $p \not\geq e+ f$ in $L$. Then $(e,f,p)$ is a distributive tiple in $L$.

**Proof:**

(i) Show $(f+p) \Lambda e = (e \Lambda f) + (e \Lambda p)$. Since $p \not\geq e$, we have $(e \Lambda f) + (e \Lambda p) = e \Lambda f$.

Suppose $f \Lambda e < (f+p) \Lambda e$. Then there exists an atom $q \leq (f+p) \Lambda e$ but $q \not\geq e \Lambda f$. Now, $q \leq f + p$ and $q \leq e$ so $q \geq 0$. By atomic exchange, we get $p \leq f + q$. But then $p \leq f + q \leq (f+q)+e = f+(q+e) = f+e+q$ since $q \leq e$. This is a contradiction.

Thus $(fp) \Lambda e = e \Lambda f$.

(ii) Show $(e+p) \Lambda f = (e \Lambda f) + (e \Lambda p)$. Since $p \not\geq e$, we have $(e \Lambda f) + (e \Lambda p) = e \Lambda f$.

Suppose $(e+p) \Lambda f < (e \Lambda f) \Lambda f$. Then there exists an atom $q \leq (e+p) \Lambda f$ but $q \not\geq e \Lambda f$. Now, $q \leq e + p$ and $q \leq f$ so $q \Lambda e$. By atomic exchange, we get $p \leq e + q$. But then $p \leq e + q \leq (e+q) + f = e + (q+f) = e + f$ since $q \leq f$.

This is a contradiction. Thus $(e+p) \Lambda f = e \Lambda f$.

(iii) Show $p \Lambda (e+f) = (p \Lambda e) + (p \Lambda f)$. But $p \Lambda (e+f) = 0 = p \Lambda e + p \Lambda f = (p \Lambda e) + (p \Lambda f)$

**Theorem 2** (Ore, 1944): Let $(L,+,\Lambda)$ be a modular atomistic lattice and $T$ a Mackey closure operator on $L$. Then $\mu (e,f)$ in $T$ if and only if $e+f$ is $T$-closed. In particular, $L_T$ is an $\mu$-symmetric lattice.

**Corollary 1:** Let $E$ be a Hausdorff topological vector space. Then for $M_{Topologically closed subspaces},$ $M+N$ is topologically closed if and only if $M$ and $N$ form a dual Modular pair in the lattice of all closed subspaces of $E$.

**Proof:**

Topological closure in a Hausdorff topological vector space is a Mackey closure operator.

Corollary: In the lattice of all closed subspaces of a real or complex normed linear spaces (Banach space, Hilbert space) $\mu(M,N)$ if and only if $M + N$ is closed.

**Linear Orthogonality Relations**

**Definition 4:** Let $(K,E)$ be a vector space over the division ring $K$, and let $\perp$ be a binary relation on $E$. We say $\perp$ is a linear orthogonality relation if and only if

i) $x \perp y$ if and only if $y \perp x$, and

ii) $(a) \ x \perp y \text{ and } x \perp z \implies x \perp y + z$

(b) $x \perp y \text{ implies } x \perp ay$ for all $x,y,z \in E$ and $a \in K$. If in addition we have

iii) $x \perp y \text{ for all } y \in E \text{ implies } x = 0$, we say $\perp$ is non-degenerate and we call the pair $(E,\perp)$, or more precisely the triple $(K,E,\perp)$, a linear orthogonality space.

For $M$ a subspace of $E$, the orthogonal of $M$ is defined by

$M^\perp = \{ x \in E | x \perp y \text{ for all } y \in M \}$

Let $P_\perp (E,\perp) = \{ F \in \text{lat}(K,E) | F = F^\perp \}$. A vector $x$ is isotropic if and only if $x \perp x$ and it is an isotropic otherwise.

**Lemma 4** (Ore, 1944): Let $\perp$ be a linear orthogonality relation on $E$ and $M$ a subset of $E$. Then

(i) $M \subseteq M^\perp$

(ii) $M \subseteq N$ implies $N^\perp \subseteq M^\perp$

(iii) $M^\perp$ is a subspace of $E$.

From Lemma 4, the map $F \mapsto F^\perp$ on the lattice of all subspaces of $E$ is a Galois autoconnection and the associated closure operator is $F \mapsto F^\perp$.

**Definition 5:** Let $\Phi$ be a $\theta$-sesquilinear form on $E$ where $\theta$ is an anti-automorphism of the underlying division ring. For $x,y \in E$, $x \perp (\Phi y)$ if and only if $\Phi(x,y) = 0$.

$(E,\perp(\Phi))$ or $(E,\Phi)$ is a linear orthogonality space if and only if $\Phi$ is a non-degenerate orthosymmetric form on $E$. In this case call $(E,\Phi)$ a quadratic space provided $E$ has at least three dimensions.

**Theorem 3** (Ore, 1944): Let $(E,\Phi)$ be a quadratic space. Then the map $F \mapsto F^\perp$, where $\perp = \perp(\Phi)$, is a normalized Mackey closure operator on $\text{lat}(K,E)$.
Proof:

From Lemma 4, we have $F \mapsto F^\perp$ is a closure operator and it is normalized since $\Phi$ is non-degenerate. Let $M = M^\perp$ and $w \in M$. Show $M + kw = (M + kw)^\perp$. Let $x \in (M + kw)^\perp$. Show $x \in M + kw$, that is $x - aw \in M$ or $x - aw \in M^\perp$ for some $a$ in $K$.

Now $w \notin (M^\perp)^{\perp\perp}$ so there is a $V$ in $M^\perp$ such that $V/1w$. We may assume $\Phi(v,w) = 1$. For $y \in M^\perp$ and $v \in M^\perp$

$$\Phi(y - \Phi(y,w)v,w) = \Phi(y,w) - \Phi(v,w)\Phi(v,w) = 0.$$ 

This implies that $y - \Phi(y,w)v$ is in $(M + kw)^\perp$.

But also $\Phi(y - \Phi(y,w)v,w) = 0$,

since $y - \Phi(y,w)v \in (M + kw)^\perp$ and $x \in (M + kw)^{\perp\perp}$.

Now

$$0 = \Phi(y - \Phi(y,w)v,x) = \Phi* + (yv) - \Phi(\Phi(v,x))$$

Thus $x - (\Phi(v,w))^\perp w$ is orthogonal to $y$ for all $y \in M^\perp$. Thus $x - (\Phi(v,w))^\perp w \in M^\perp$.

The following theorem answers the question, “which linear orthogonality spaces are quadratic spaces.

**Theorem 4** (Ore, 1994): Let $(E,L)$ be a linear orthogonality space of dimension at least 3 such that $M \mapsto M^\perp$ is a Mackey closure operator on $\text{lat}(K,E)$. Then there exists a nondegenerate $\theta$-sesquilinear form $\Phi$ on $E$ such that $x \perp y$ if and only if $\Phi(x,y) = 0$.

**Proof:**

Let $[x] = kv$. First note that $[x]^\perp = [x]^\perp$ is a hyperplane, so $[x]^\perp = \ker(f_0)$.

for some linear functional $f_0$.

Define $\varphi : \text{line}(E) \to \text{lines}(E^\perp)$ by $\varphi(x) = [f_0] \to$, where $[x]^\perp = \ker(f_0)$.

$\varphi$ is well defined since if $x = y$, then $[x]^\perp = [y]^\perp$ so $\ker(f_0) = \ker(f_0)$.

Thus $f_0 = af_0$ so $[f_0] = [f_0]$ whence $\varphi(x) = \varphi(y)$. Also $\varphi$ is one-one since if $\varphi(x) = \varphi(y)$ then $\ker(f_0) = \ker(f_0)$ so $[x]^\perp = [y]^\perp$ which implies $x = y$ since for a Mackey closure operator, finite dimensional subspaces are closed. Now if $[x_0] \subseteq [x_1] + [x_2]$ then $[x_1]^\perp \cap [x_2]^\perp \subseteq [x_0]^\perp$ so that $\ker(f_0) \cap$

$\ker(f_0) \subseteq \ker(f_0)$.

Therefore $f_0 = af_0 + \beta f_2 \Rightarrow [f_0] \subseteq [f_1] + [f_2]$ or $\varphi(x_0) \subseteq \varphi(x_1) + \varphi(x_2)$.

In general, $\varphi$ is not onto unless $E$ is finite-dimensional. So let $F$ be a subspace of $E^\perp$ generated by $im(\varphi)$. We shall show that $\varphi$ is onto the lines of $F$.

Let $[g]$ be a line in $F$. Then $g = \sum_{i=1}^n \alpha_i f_x^i \Rightarrow \ker(g) = \cap_{i=1}^n \ker(f_x^i) + M$.

where $M$ is finite-dimensional.

Again since we have a Mackey closure operator, adding, the finite-dimensional subspace $M$ to the closed subspace $\cap_{i=1}^n ker(f_x^i)$

yields a closed subspace, namely $\ker(g)$. Then $\ker(g) = (\ker(g))^\perp \subseteq [x]^\perp$ which is a hyperplane. Thus

$\ker(g) = [x]^\perp \Rightarrow \ker(f_2)$.

$\text{org} = af_0$ or $[g] = [f_2]$.

Applying the Fundamental Theorem of Projective Geometry, we get a semilinear bijection $T : E \to F$ with respect to an automorphism $\theta$ of $K$ such that $\varphi(x) = [T_x]$. Note $T(\lambda x) = T(x)\lambda^0$ since $E$ is a left and $F \subseteq E$ is a right vector space.

**Define**

$\Phi : E \times E \to K$

by

$\Phi(x,y) = T(y)(x)$

$\Phi$ is linear in $x$ and $\theta$-linear in $y$. Also, $\Phi$ is non-degenerate since $T$ is one-one. Moreover $\Phi(x,y) = 0$ if and only if $T(y)(x) = 0$ if and only if $x \in \ker(T(y)) = \ker(f_0) = [y]^\perp$ if and only if $x \perp y$.

**Corollary:** Let $(E,L)$ be a linear orthogonality space of dimension at least 3. Then $(E,L)$ is a quadratic space if and only if the closure operator induced by $\perp$ on $\text{lat}(K,E)$ is a Mackey closure operator.

**Proof:**
Assuming $(E, \perp)$ is a quadratic space, then from the remark under Lemma 4, $F \mapsto F \perp$ is a Mackey closure operator.

Conversely, let $F \mapsto F \perp$ be a Mackey closure operator induced by $\perp$ on $\text{lat}(K, E)$. By Theorem 4, there exists a non-degenerate $\theta$ sesquilinear form $\Phi$ on $E$ such that $\lambda \perp y \mapsto f$ and only if $\Phi(x, y) = 0$ and it is normalised since $\Phi$ is non-degenerate. It follows from Theorem 3 that $(E, \perp)$ is a quadratic space.

The next result is a generalization to infinite dimension of the result of (Birkhoff and von Neumann, 1936).

**Theorem 5:** Let $\perp : \text{lat}(K, E) \to \text{lat}(K, E)$ be an orthocomplementation. Then $(K, E)$ is finite-dimensional and there exists a nondegenerate $\theta$ sesquilinear form $\Phi$ admitting no non-zero isotropic vectors such that $M = \{ x \in E | \Phi(x, M) = 0 \text{ for all } m \in M \}$.

**Proof:** Since $\perp : \text{lat}(K, E) \to \text{lat}(K, E)$ is an orthocomplementation, then the map $F \mapsto F \perp$ is a Galois autoconnection and the associated closure operator is $F \mapsto F \perp$. There exists a non-degenerate $\theta$ sesquilinear form $\Phi$ on $E$ such that $(E, \Phi)$ is a quadratic space and the map $F \mapsto F \perp$, where $\perp = \perp(\Phi)$, is an associated closure operator on $\text{lat}(K, E)$. It is clear that $(E, \perp(\Phi)) = (E, \Phi)$, is a linear orthogonality space if and only if $\Phi$ is a non-degenerate orthosymmetric form on $E$.

By (Theorem 4) $x \perp y$ if and only if $\Phi(x, y) = 0$. Thus, $M \in E \implies M' = \{ x \in E | x \perp y \text{ for all } m \in M \}$ if and only if $M = \{ x \in E | \Phi(x, y) = 0 \text{ for all } m \in M \}$.

**Orthogonality and Locally Convex Topologies**

Let $(E, \perp)$ be a linear orthogonality space. If $(E, \perp)$ has a vector space topology, which gives the same closed subspaces as the linear orthogonality relation, then there exists always a locally convex topology with the same property.

Those topologies which give the same closed subspaces as the linear orthogonality relation have been characterized by (Kakol and Sorjonen, 1991) with the aid of Hahn-Banach property. This, in a special case, gives us a necessary and sufficient condition for a topology to be locally convex. It also provides a means for an extension of a theorem of (Kakutani and Mackey, 1944, 1946).

**Basic Definitions**

**Definition 6**

(i) $E$ is an infinite dimensional vector space over field $K$, which is either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$.

(ii) $(E, \tau)$ is a Hausdorff topological vector space

(iii) $E'$ is the algebraic dual of $E$ and $E'$ or $(E, \tau)$ is its topological dual.

(iv) $\mathcal{L}(E)$ is the set of all r-closed vector subspaces of $E$.

(v) The orthogonal $X'$ of a set $X$ in $E$ is the subspace $\{ y \in E | \exists x \in X \}$.

(vi) A vector subspace $M$ of $E$ is orthoclosed, if $M = M \perp$.

(vii) $(E, \perp)$ is the orthodual of a linear orthogonality space $E$ consisting of orthocontinuous linear functionals $\phi$, i.e. $\phi \in E'$ and the ker $\phi$ is orthoclosed.

**Definition 7** (Hahn–Banach separation property HBSP) Let $(E, \perp)$ be a linear orthogonality space and let $L_{\perp}(E)$ be the lattice formed by orthoclosed subspaces of $E$. Then $M \in L_{\perp}(E)$ and $x \in M \implies$ there exists $f \in (E, \perp)^0$ s.t. ker $f \supset$ ker $M$ and $x \in$ ker $f$.

**Definition 8 (Orthocomplementation):**

An orthocomplementation on $L(E)$ is a mapping on $L(E)$ with the properties:

1. $M \subseteq \mathcal{N}$ implies $M' \supseteq \mathcal{N}'$
2. $M' = M$ for all $M \in L(E)$
3. $M \cap M' = \{0\}$ for all $M \in L(E)$

**Existence of a Locally Convex Topology**

**Theorem 6** (Kakol and Sorjonen, 1991): Let $(E, \perp)$ be a linear orthogonality space endowed with Hausdorff vector space topology, $\tau$. If $L_{\perp}(E) = L_{\perp}(E)$, then there exists a locally convex space topology $\sigma$ on $E$ such that $L_{\sigma}(E) = L_{\sigma}(E)$.

**Proof:**

$L_{\sigma}(E) = L_{\perp}(E) \Rightarrow (E, \tau)' = (E, \perp)'$ and that the relation $\perp$ has the Mackey property $M \in L_{\perp}(E)$ and $x \in E \Rightarrow M + hxiE_{\perp}(E)$.
hereby denotes the vector subspace spanned by \( \{ \cdot \} \). This in turn implies that the space \((E, \perp)\) is a quadratic space i.e., there exists an automorphism \(V\) of the field \(K\) and a non-degenerate orthosymmetric \(V\)-sesquilinear form \([\cdot, \cdot]\) on \(E\). Furthermore, \([x, y] = 0\) if and only if \(x \perp y = 0\); see (previous section).

The Frechet-Riez representation theorem implies that the orthodual \((E, \perp)\) consists of the functions of the form \([\cdot, x]\) with \(x \in E\). This together with the relation \((E, \perp)^0 = (E, \tau) = E^\prime\) means that \((E, E^\prime)\) is a dual pair. The weak topology \(\alpha\) induced by this duality is a locally convex space topology on \(E\) such that \((E, \alpha) = (E, \tau)\).

As the weak topology \(\alpha\) is the coarsest topology on \(E\) for which the forms \(f \in (E, \tau)'\) and continues, we have the relation \(L(\alpha) \subset L(\tau)\). Thus \(L(\alpha) = L(\tau)\).

To prove the opposite inclusion, note first that the subspaces of the form \(\{x\}\) are ortholoced hyperplanes which in turn are all of the form \(\ker f \in (E, \perp)\). Using this, we get for all subspaces \(M \in L(\alpha) = L(\tau)\).

\[ M = M_{\perp, \perp} = \left( \bigcup_{x \in M^\perp} \{x\} \right)^\perp = \bigcap_{x \in M^\perp} \{x\} \cap \bigcap_{x \in M^\perp} \ker f, \]

because \(f \in (E, \tau)' = (E, \alpha)'\). Thus \(L(\alpha) \supset L(\perp)\)

\[ \star \]

**Corollary 4**: Let \((E, \tau)\) be a topological vector space. Suppose that \(L(\tau)\) admits an orthocomplementation. Then there exists a locally convex space topology \(\alpha\) on \(E\) such that \(L(\alpha) = L(\tau)\).

**Proof**.

Let us define a relation \(\perp\) by the rule \(x \perp y\) for all \(x \in E\):

(i) \(x \perp y \Rightarrow (y) \subset (x) \Rightarrow (x) \Rightarrow (y)\)

i.e. \((y) \subset (x) \Rightarrow y \perp x\)

(ii) \((x) \cap (x) = \{0\}\); using (Definition 8 (3))

i.e. \(\{x\} \cap \{x\} = \{0\} \Rightarrow \{x\} = \{x\} \cap \{x\} \forall x \in E\)

We need to show that \(L(\alpha) = L(\perp)\).

\[ M \in L(\alpha) \Rightarrow M = M^{\perp} = M^{\perp, \perp} \Rightarrow M \in L(\perp) \]

Thus \(L(\alpha) \subset L(\perp)\).

Conversely, suppose \(M \in L(\perp)\). Then \(M = M^{\perp} = M \Rightarrow M \in L(\alpha)\). Thus \(L(\alpha) \subset L(\perp)\). It follows that \(L(\alpha) = L(\perp)\). By (Theorem 6), there exists a locally convex space topology \(\alpha\) on \(E\) such that \(L(\alpha) = L(\tau)\).

**Theorem 3.3** (of Ward, (1942)) can be stated as follows:

**Theorem 7**: If \(L(\alpha)\) admits an orthocomplementation, then there exists a locally convex topology on \(E\) such that \(L(\alpha) = L(\tau)\).

**Characterization of Locally Convex Topologies**

Let \((E, \perp)\) be a linear orthogonality space and let \(\tau\) be a vector space topology on \(E\). Under what conditions is \(\tau\) considered a locally convex topology?

**Theorem 8**: If \((E, \perp)\) is a linear orthogonality space which has the Hahn-Banach separation property (HBSP), and \(\tau\) is a Hausdorff vector space topology on \(E\) such that \((E, \tau)' = (E, \perp)\), then \(L(\tau) = L(\perp)\), if and only if the topology \(\tau\) has the Hahn-Banach separation property (HBSP).

**Proof**.

For all \(M \in L(\tau) = L(\perp)\)

\[ M = M^{\perp, \perp} = \left( \bigcup_{x \in M^\perp} \{x\} \right)^\perp = \bigcap_{x \in M^\perp} \{x\} \cap \bigcap_{x \in M^\perp} \ker f_x \]

and \(f \in (E, \perp)' = (E, \tau)' \) i.e. \(f \in (E, \tau)'\). This implies that \(\ker f_x \supset M\) and \(x \notin \ker f\), i.e. \(x \notin M\).

This shows that the topology \(\tau\) has the Hahn-Banach separation property.

Conversely, suppose \(\tau\) has the Hahn-Banach separation property. Then \(M \in L(\tau)\) and \(x \notin M \Rightarrow \exists f \in (E, \tau)\)

s.t. \(\ker f \supset M\) and \(x \notin \ker f\). Since \(f \in (E, \tau)' = (E, \perp)\) and \((E, \perp)\) has the Hahn-Banach separation property, then \(M \in L(\perp)\). Thus

\(L(\tau) \subset L(\perp)\).

Also,

\(M \in L(\perp)\) and \(x \notin M \Rightarrow \exists f \in (E, \perp)' \ker f \supset M\) and \(x \notin \ker f \Rightarrow M \in L(\tau)\).

Thus \(L(\perp) \subset L(\tau)\). Therefore \(L(\tau) = L(\perp)\).

\[ \star \]
Corollary 5: In addition to the assumption of (Theorem 8), if the topology \( \tau \) is metrizable and complete, then \( \tau \) is locally convex, if and only if \( L(E) = L_{1,1}(E) \).

Corollary 6: Suppose a linear orthogonality space \((E, \perp)\) is also a \(p\)-Banach space with \(0 < p \leq 1\), and \( \tau \) is the topology induced by the \(p\)-norm. If the linear orthogonality relation \( \perp \) is definite, i.e., \( x \perp y \) implies \( x = 0 \), and \( L(E) = L_{1,1}(E) \), then \( E \) is a Hilbert space with the natural norm equivalent to the \(p\)-norm of \( E \).

The following result is an extension of a theorem of (Kakutani and Mackey, 1944)

Corollary 7: Let \( E \) be a \(p\)-Banach space with \(0 < p \leq 1\), and let \( \tau \) be the topology induced by the \(p\)-norm. If \( L(E) \) admits an orthocomplementation, then \( E \) is a Hilbert space with the natural norm equivalent to the \(p\)-norm of \( E \), and the orthocomplementation is the usual one.

Conclusion

In this paper, we have brought out the different characterizations and applications of Mackey closure operators and locally convex topologies. In particular, the notion of linear orthogonality space which can be extended to a more general form.

The notion of linear orthogonality space can be seen in a more general form if we define it as follows:

A pair \((E, \perp)\) is a linear orthogonality space provided \( E \) is a real linear space with \( \dim E \geq 2 \) and \( \perp \subseteq E^2 \) is a relation such that

\( (L01) \) \( x \perp y \) if and only if \( y \perp x \), \( x \perp y \) for every \( x \in E \) implies \( y = 0 \), or \( y \perp x \) for all \( y \in E \) implies \( x = 0 \)

\( (L02) \) if \( x, y \in E \setminus \{0\} \) and \( x \perp y \), then \( x \) and \( y \) are linearly independent,

\( (L03) \) if \( x, y, z \in E \), \( x \perp y \) and \( x \perp z \) imply \( x \perp y + z \)

\( (L04) \) if \( x, y \in E \) and \( x \perp y \), then \( ax \perp by \) for every \( a, b \in \mathbb{R} \),

\( (L05) \) if \( P \) is a 2-dimensional subspace of \( E \), \( x \in P \) and \( a \in \mathbb{R} \), then there is \( y \in P \) with \( x \perp y \) and \( x + y \perp ax - y \).

A linear space can be made into a linear orthogonality space if we define \( x \perp 0 \), \( 0 \perp x \) for all \( x \), and for non-zero vectors \( x, y \), define \( x \perp y \) if \( x, y \) are linearly independent.


